PERGAMON

International Journal of Solids and Structures 37 (2000) 2709–2726

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Optimal design of elastic rods under axial gravitational load using the maximum principle

William H. Warner*

Department of Aerospace Engineering and Mechanics, University of Minnesota, 107 Akerman Hall, 110 Union Street S.E., Minneapolis, MN, 55455, USA

Received 17 July 1998; in revised form 10 February 1999

Abstract

Optimal design problems for structural elements in equilibrium often come in pairs in which the cost function for one problem becomes a constraint for the second. In particular, the problem of minimizing structural volume or weight under size and stress constraints has a dual in which potential energy is minimized for fixed volume of material. These dual problems are solved here for a linearly elastic rod hanging from a rigid support. The design variable is the cross-sectional area of the rod.

The method used is the Maximum Principle of Pontryagin and Hestenes, rather than the function space methods used by others for similar problems.

Keywords: Optimal design; Elastic rods; Variational methods; Maximum principle

1. Two optimal design problems for elastic rods under axial gravitational load

Optimal design problems for structural elements in equilibrium often come in pairs in which the cost function for one problem becomes a constraint for the second. In particular, the problem of minimizing structural volume or weight under size and stress constraints has a dual in which potential energy is minimized for fixed volume of material. These dual problems are solved here for a linearly elastic rod hanging from a rigid support. The design variable is the cross-sectional area of the rod.

The equilibrium problem for given area distribution is easy to state and easy to solve formally. The undeformed length L, Young's modulus E, weight per unit volume ρg, and a piecewise continuous distribution of cross-sectional area $A(X)$ are given, where X is the vertically downward coordinate axis through the centroids of the cross sections. The loads are the rod's weight per unit length and the

* Corresponding author. Fax: +1-612-626-1558.
E-mail address: warner@aem.umn.edu (W.H. Warner).

0020-7683/00/$ - see front matter
PII: S0020-7683(99)00045-1

(unknown) end forces necessary to maintain the displacement of the lower end at a known value Δ. The axial displacement $U(X)$ and the axial force $P(X)$, both continuous functions, are found together with the axial strain $\epsilon(X)$ from the equations

$$\frac{\mathrm{d}U}{\mathrm{d}X} = \epsilon(X),$$

$$P(X) = EA(X)\epsilon(X),$$

$$\frac{\mathrm{d}P}{\mathrm{d}X} + \rho g A(X) = 0. \tag{1a}$$

These hold at all points of continuity of $A(X)$ in $0 < X < L$, with $\epsilon(X)$ discontinuous where A is. The displacement U satisfies boundary conditions

$$U(0) = 0, \quad U(L) = \Delta. \tag{1b}$$

This problem has the formal solution

$$P(X) = EA(X)\epsilon(X) = P(0) - \rho g \int_0^X A(Y)\mathrm{d}Y$$

$$U(X) = \frac{P(0)}{E}\int_0^X \frac{\mathrm{d}Y}{A(Y)} - \frac{\rho g}{E}\int_0^X \frac{1}{A(Y)}\int_0^Y A(Z)\mathrm{d}Z\,\mathrm{d}Y \tag{2a}$$

with the value of $P(0)$ determined by the boundary condition $U(L) = \Delta$:

$$P(0) = \frac{E\Delta}{\int_0^L \frac{\mathrm{d}Y}{A(Y)}} + \frac{\rho g \int_0^L \frac{1}{A(Y)}\int_0^Y A(Z)\mathrm{d}Z\,\mathrm{d}Y}{\int_0^L \frac{\mathrm{d}Y}{A(Y)}}. \tag{2b}$$

Note that the solution (2) scales in a special way with $A(X)$: the axial force P scales linearly with the area but U, ϵ, and $\sigma = E\epsilon$ remain unchanged. That is, if $\{P, U, \epsilon, \sigma\}$ are the solutions for the choice $A(X)$, then the solutions $\{P^*, U^*, \epsilon^*, \sigma^*\}$ for the choice $A^*(X) = KA(X)$ will be $\{KP, U, \epsilon, \sigma\}$.

The solution is equally well characterized by the Principle of Minimum Potential Energy. The second-order differential equation governing the displacement at equilibrium then appears as the Euler–Lagrange equation for the functional

$$\Pi(U(X)) = \int_0^L A(X)\left\{\frac{1}{2}E\epsilon^2(X) - \rho g U(X)\right\}\mathrm{d}X \tag{3}$$

assuring a stationary value for Π among all displacement fields satisfying the displacement boundary conditions (1b) and with $\epsilon = \mathrm{d}U/\mathrm{d}X$. Here $A(X)$ is regarded as given.

Two optimal design problems for such a column are solved here. The design variable is the section area $A(X)$. The first design problem is the minimization of the total volume

$$V = \int_0^L A(X)\mathrm{d}X \tag{4}$$

of a column fixed at both ends and in equilibrium under its own weight [Eqs. (1) as constraints with $\Delta = 0$] and subject to inequality constraints on both section size and axial stress $\sigma(X) = P(X)/A(X)$:

$$A_1 \le A(X) \le A_2, \quad |\sigma(X)| \le \sigma_0. \tag{5}$$

The second problem is the minimization of Π [Eq. (3)] considered as a functional of U, A and ϵ subject to the differential equation constraint

$$\frac{\mathrm{d}U}{\mathrm{d}X} = \epsilon(X) \tag{6}$$

from the set (1a), the boundary conditions (1b) on U, the inequality constraints

$$A_1 \le A(X) \le A_2 \tag{7}$$

on section size and the integral ('isoperimetric') equality constraint of prescribed total volume:

$$\int_0^L A(X)\mathrm{d}X = V_0. \tag{8}$$

We must have $A_1 L \le V_0 \le A_2 L$ for consistency.

The first design problem involves a loading case not treated by Velte and Villaggio (1982). They solve the problem of finding the distribution of cross-sectional area that minimizes the total volume of a linearly elastic rod fixed at its ends and loaded axially with a known load along its length. Bounds on section size and on axial stress are prescribed. By constructing minimizing sequences in the space of L^2 functions, they show that the problem has at least one solution if the set of admissible designs satisfying the constraints is nonempty. They show that no, one, or infinitely many solutions may exist for a rod loaded with a concentrated force at its midpoint, depending on the choice of constraint parameters.

The indefinite integral of the given axial load plays a critical part in their argument. However, if the axial load arises from the weight on the bar itself, then it and its integral will also depend on the design variable. That is the problem considered here.

The second design problem is analogous to the problem solved by Fosdick and Royer-Carfagni (1996) (see also Fosdick et al., 1996) for two-phase mixtures in stressed bars. By replacing $A(X)$ by a different design variable $c(X)$ where

$$A(X) = A_2 c(X) + A_1(1 - c(X)), \quad 0 \le c(X) \le 1,$$

$A(X)$ takes on the form of the density ρ in Fosdick and Royer-Carfagni (1996), Eq. (2.2). The methods used there could then be followed and would indicate that discontinuous solutions occur with two or three subregions in which A takes on its extreme values alternately, depending on the parameters. As stated in Fosdick and Royer-Carfagni (1996), the essential difficulty arises because of the nonconvexity of the strain energy density function $(1/2)EA\epsilon^2$ in the pair of quantities ϵ and c, where the stiffness function $S = EA$ depends on c. Whereas they treat this difficulty by constructing a related problem having the same minimizer from the lower convex envelope of the strain energy density, here the solution is found by direct computation utilizing the Maximum Principle (Pontryagin et al., 1962; Hestenes, 1980). The same method will be used for the solution of the minimum volume problem.

Section 2 applies the Maximum Principle to the minimum volume problem and summarizes the solution procedure. Section 3 does the same for the minimum potential energy problem. Most of the details of the computations are relegated to Appendices. Similar results for Euler–Bernouilli beams will be reported in a separate paper.

2. The minimum volume problem for the rod under gravity

Part of the solution of the minimum problem [Eqs. (1–5)] for the fixed–fixed rod requires little computation. If a lower bound A_1 on section size is imposed on admissible designs, then $A(X)$ identically equal to A_1 will give the minimum attainable volume A_1L and will satisfy trivially any upper bound constraint on A. Because of the scaling property noted after Eq. (2), all uniform sections give the same stress distribution when $\Delta = 0$:

$$\sigma(X) = \frac{\rho g L}{2}\left(1 - \frac{2X}{L}\right).$$

Maximum stress magnitude occurs at the ends and is $(\rho g L)/2$. Therefore the uniform section A_1 is the unique optimizer as long as the rod is short enough:

$$L \leq \frac{2\sigma_0}{\rho g}. \tag{9}$$

(I note parenthetically that for even a low value of allowable stress such as $\sigma_0 = 70$ MPa for a heavy material such as steel ($\rho g = 77$ kN/m^3) the limiting length is approximately 1.8 km or 1.1 miles. The engineer's phrase 'For all practical purposes ...' would seem to be appropriate here.)

When (9) is violated the solution form changes to a three-region solution for $A(x)$ consisting of exponentially decaying and growing segments at top and bottom symmetrically placed about a central segment at minimum size. This solution is valid until the end sections reach maximum allowable size at

$$\frac{\rho g L}{\sigma_0} = 2\left[1 + \ln\left(\frac{A_2}{A_1}\right)\right].$$

No solution exists at greater values of the parameter $\rho g L/\sigma_0$ since no design exists satisfying all constraints.

The problem is more easily discussed in nondimensional form. Set

$$X = Lx, \quad A(X) = A_1 a(x), \quad U(X) = \frac{L\sigma_0}{E}u(x),$$

$$\epsilon(X) = \frac{\sigma_0}{E}\eta(x), \quad \sigma(X) = \sigma_0\tau(x), \quad P(X) = A_1\sigma_0 p(x).$$

Then $\tau(x) = \eta(x)$ and the separate symbol for the nondimensional stress is not needed.

The optimization problem becomes the minimization of

$$\int_0^1 a(x)\mathrm{d}x \tag{10}$$

subject to

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \eta(x), \quad \frac{\mathrm{d}p}{\mathrm{d}x} = -\gamma a(x), \tag{11a}$$

$$u(0) = u(1) = 0, \tag{11b}$$

$$\phi_1 \equiv 1 - a \leq 0, \tag{11c}$$

$$\phi_2 \equiv a - \hat{a} \leq 0, \tag{11d}$$

$$\phi_3 \equiv -1 - \eta \leq 0, \tag{11e}$$

$$\phi_4 \equiv \eta - 1 \leq 0, \tag{11f}$$

$$\phi_5 \equiv a\eta - p = 0. \tag{11g}$$

Here $\gamma = (\rho g L)/\sigma_0$ and $\hat{a} = A_2/A_1$.

The mathematical formalism showing that difficulties may be expected here requires the computation of the minors of order 5 of the 5-by-6 regularity matrix for the constraints [Hestenes, 1980, p. 260, Eq. (4.2)]. Its six minors of order 5 are all zero if one of ϕ_1, ϕ_2 equals zero at the same location that one ϕ_3, ϕ_4 does. Under these conditions a design satisfying all constraints may not exist and Velte and Villaggio's condition for a well-posed problem may not be met.

Return to the problem of minimizing (2) subject to (3). The necessary conditions in addition to (3) are obtained by applying the Maximum Principle (Hestenes, 1980, Chapter 6, Theorem 4.1) for 'control problems of Lagrange with inequality constraints'. There are two state variables $u(x)$ and $p(x)$ and two controls $a(x)$ and $\eta(x)$. If a minimizer $\{u^*(x), p^*(x), \eta^*(x), a^*(x)\}$ exists with u^* and p^* continuous with piecewise continuous derivatives and h^*, a^* piecewise continuous on [0,1], then there exist multipliers

$$\lambda_0 \geq 0, \quad v(x), \quad q(x), \quad \mu_\alpha(x) \quad (\alpha = 1, 2, 3, 4, 5) \tag{12a}$$

not vanishing simultaneously on the closed interval $0 \leq x \leq 1$ and a 'pre-Hamiltonian' function $H(u, p, v, q, a, \eta, \mu_\alpha)$ defined by

$$H(u, p, v, q, a, \eta, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5) = v\eta + q(-\gamma a) - \lambda_0 a - \sum_{i=1}^{5} \mu_i \phi_i \tag{12b}$$

such that the following conditions hold.

1. The solution multipliers $\mu_\alpha(x)$ are continuous on each interval of continuity of $\eta^*(x)$, $a^*(x)$. Moreover, they are nonnegative for $\alpha = 1, 2, 3, 4$ with each of these four equal to zero wherever the corresponding ϕ_α function [Eqs. (11c)–(11f)] is strictly less than zero.
2. The solution multipliers $v^*(x)$ and $q^*(x)$ are continuous and satisfy, with $u^*(x)$, $p^*(x)$, $\eta^*(x)$, $a^*(x)$ and the five μ's, the given differential equation constraints in (11a) repeated here

$$\frac{du}{dx} = \frac{\partial H}{\partial v} = \eta(x), \quad \frac{dp}{dx} = \frac{\partial H}{\partial q} = -\gamma a \tag{13a}$$

and the additional equations

$$\frac{dv}{dx} = -\frac{\partial H}{\partial u} = 0, \quad \frac{dq}{dx} = -\frac{\partial H}{\partial p} = \mu_5(x) \tag{13b}$$

$$\frac{\partial H}{\partial a} = -\gamma q(x) - \lambda_0 + \mu_1 - \mu_2 - \eta\mu_5 = 0 \tag{13c}$$

$$\frac{\partial H}{\partial \eta} = v(x) + \mu_3 - \mu_4 - a\mu_5 = 0 \tag{13d}$$

on each interval of continuity of $\eta^*(x)$, $a^*(x)$. The equality constraint (11g) must also hold.

3. The function of H of (12b) evaluated on a minimizer is continuous on the closed interval $0 \leq x \leq 1$ and satisfies the relation

$$\frac{\mathrm{d}H}{\mathrm{d}x} = \frac{\partial H}{\partial x} = 0 \tag{14}$$

on each interval of continuity of $\eta^*(x)$, $a^*(x)$.

4. The inequality

$$H(u^*, p^*, v^*, q^*, q, \eta, 0, 0, 0, 0, 0) = v^*\eta - (\gamma q^* + \lambda_0)a \leq \bar{H} = \sup(H) \tag{15}$$

holds for all admissible controls $\eta(x)$ and $a(x)$ on the left side and for the solution controls $\eta^*(x)$, $a^*(x)$ on the right together with the solution values of $u^*(x)$, $p^*(x)$, and the multipliers $v^*(x)$, $q^*(x)$, λ_0, λ on both sides of (15). The constant $\bar{H}$ is the value of the supremum of H guaranteed by the result Eq. (14) since the H function does not depend explicitly on x.

Since the boundary values on $u(x)$ are given at both ends of the rod, there are no boundary conditions on $v(x)$ arising from the standard transversality relations ('natural boundary conditions'). However, transversality requires

$$q(0) = q(1) = 0 \tag{16}$$

since p is not prescribed at the ends.

The multiplier functions $\mu_1(x)$ and $\mu_2(x)$ cannot both be positive since both functions $\phi_{1,2}$ cannot be zero simultaneously; similarly both $\mu_3(x)$ and $\mu_4(x)$ cannot both be positive since $\phi_{3,4}$ cannot be zero simultaneously. The conditions $\partial H/\partial a = \partial H/\partial \eta = 0$ [Eqs. (13c), (13d)] determine the values of the nonzero $\mu_\alpha(x)$, $\alpha = 1, 2, 3, 4$, once the other quantities have been found.

Since the solutions $u^*(x)$, etc., are not known a priori to use in the Maximum-H inequality (15), the function on the left of (15) is replaced by

$$\tilde{H}(a, \eta) = v\eta - (\gamma q + 1)a. \tag{17}$$

Its supremum will be found with respect to the control variables regarding $v(x)$ and $q(x)$ as known. This procedure only determines combinations that give a local supremum and some other condition must be used to test the candidate for possible global optimality.

Five steps are involved in generating the optimizers. First, the constant value of $v(x)$ is shown to be zero [Appendix A(1)]. This involves the calculation of sup $\tilde{H}$ with respect to η and use of the fact that $|\eta| = 1$ cannot hold on all of [0, 1]. Second, the value 0 for the nonnegative scalar Lagrange multiplier λ_0 is ruled out by showing that all multipliers would then be zero simultaneously [Appendix A(2)]. Take $\lambda_0 = 1$ from here on. Third, return to finding sup $\tilde{H}$ with respect to $a(x)$ to show that the optimal $a(x)$ and so the optimal $\eta(x)$ must be continuous [Appendix A(3)]. Fourth, the case when $a(x)$ is constant for the whole rod is solved [Appendix A(4)], agreeing with the elementary result discussed at the beginning of this Section when $a(x)$ is at its minimum value. Finally, in Appendix A(5), the case when $|\eta| = 1$ over part of the interval is treated leading to the optimal design

$$a^*(x) = \begin{cases} \exp\left[-\gamma\left(x-\frac{1}{2}\right)-1\right], & 0 \le x \le \frac{1}{2}-\frac{1}{\gamma}; \\ 1, & \frac{1}{2}-\frac{1}{\gamma} \le x \le \frac{1}{2}+\frac{1}{\gamma}; \\ \exp\left[+\gamma\left(x-\frac{1}{2}\right)-1\right], & \frac{1}{2}+\frac{1}{\gamma} \le x \le 1 \end{cases} \tag{18}$$

valid for $2 \le \gamma \le 2+2\ln(\hat{a})$. The upper bound is the value of γ where $a^*(0)$ and $a^*(1)$ become $\hat{a}$.

3. Minimum potential energy for fixed volume

The second design problem is the minimization of the potential energy functional (3) subject to the constraints (6)–(8). This problem is also more easily treated if it is cast in a nondimensional formulation. Replace the quantities $\{X, A, U, \Delta, \epsilon, \Pi\}$ by $\{x, a, u, \delta, \eta, \pi\}$, respectively, where

$$X = Lx, \quad A(X) = \hat{A}a(x),$$

$$U(X) = \frac{\rho g L^2}{E}u(x), \quad A(X)\Delta = \frac{\rho g L^2}{E}\delta,$$

$$\epsilon(X) = \frac{\rho g L^2}{E}\eta(x), \quad A(X)\Pi = \frac{\rho^2 g^2 L^3 \hat{A}}{E}\pi.$$

The quantity $\hat{A}$ used to nondimensionalize the area function may be taken to be one of the bounds A_1, A_2 or the constant area $A_0 = V_0/L$ that gives the prescribed volume; the most useful choice seems to be a weighted harmonic mean of the bounds:

$$\hat{A}^{-1} = \xi_1 A_1^{-1} + \xi_2 A_2^{-1},$$

$$\xi_1 = \frac{A_2 - A_0}{A_2 - A_1}, \quad \xi_2 = \frac{A_0 - A_1}{A_2 - A_1}. \tag{19}$$

This choice is suggested by the solution process itself and is another connection to Fosdick and Royer-Carfagni (1996), where such a choice occurs in the construction of the lower convex envelope of the strain energy density.

The design problem is then the minimization of

$$\pi(u, a, \eta) = \int_0^1 a(x)\left(\frac{1}{2}\eta^2(x) - u(x)\right)\mathrm{d}x \tag{20}$$

with respect to u, a, and η subject to the constraints

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \eta(x), \tag{21a}$$

$$u(0) = 0, \quad u(1) = \delta, \tag{21b}$$

$$\int_0^1 a(x)\mathrm{d}x = a_0, \tag{21c}$$

$$\phi_1 \equiv a_1 - a \leq 0, \quad \phi_2 \equiv a - a_2 \leq 0 \tag{21d}$$

where $\{a_0, a_1, a_2\} = \{A_0, A_1, A_2\}/\hat{A}$, $0 < a_1 < a_0 < a_2$, and using (19)

$$\xi_1 a_1^{-1} + \xi_2 a_2^{-1} = 1$$

$$\xi_1 = \frac{a_2 - a_0}{a_2 - a_1}, \quad \xi_2 = \frac{a_0 - a_1}{a_2 - a_1}. \tag{21e}$$

The regularity condition [Hestenes, 1980, p. 260, Eq. (4.2)] on the constraint functions ϕ_k is satisfied since the nontrivial 2-by-2 minors of the 2-by-4 regularity matrix cannot all be zero simultaneously.

This problem can also be solved using the Maximum Principle. The state function is $u(x)$ and the controls are $\eta(x)$ and $a(x)$. If there exists a minimizer $\{u^*(x), \eta^*(x), a^*(x)\}$ of the functional (20) satisfying the constraints (21) with u^* continuous with piecewise continuous derivatives and η^*, a^* piecewise continuous on [0, 1], then there exist multipliers

$$\lambda_0 \geq 0, \quad \lambda, \quad p(x), \quad \mu_1(x), \quad \mu_2(x) \tag{22a}$$

not vanishing simultaneously on the closed interval $0 \leq x \leq 1$ and a 'pre-Hamiltonian' function $H(u, p, \eta, a, \mu_1, \mu_2)$ defined by

$$H = p\eta - \lambda_0 a\left(\tfrac{1}{2}\eta^2 - u\right) - \lambda a - \mu_1\phi_1 - \mu_2\phi_2 \tag{22b}$$

such that the following conditions hold.

1. The multipliers $\mu_1(x)$, $\mu_2(x)$ are continuous on each interval of continuity of $\eta^*(x)$, $a^*(x)$. Moreover, they are nonnegative. Each is zero wherever the corresponding ϕ_α function Eq. (21) is strictly less than zero.
2. The multiplier $p(x)$ is continuous and satisfies, with $u^*(x)$, $\eta^*(x)$, $a^*(x)$ and the two μ's, the given differential equation constraint (21)

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\partial H}{\partial p} = \eta(x) \tag{23a}$$

and the additional equations

$$\frac{\mathrm{d}p}{\mathrm{d}x} = -\frac{\partial H}{\partial u} = -\lambda_0 a(x) \tag{23b}$$

$$\frac{\partial H}{\partial \eta} = p - \lambda_0 a\eta = 0 \tag{23c}$$

$$\frac{\partial H}{\partial a} = \mu_1 - \mu_2 - \lambda - \lambda_0\left(\frac{1}{2}\eta^2 - u\right) = 0 \tag{23d}$$

on each interval of continuity of $\eta^*(x)$, $a^*(x)$.

3. The function H evaluated on a minimizer is continuous on the closed interval $0 \leq x \leq 1$ and satisfies the relation

$$\frac{dH}{dx} = \frac{\partial H}{\partial x} = 0 \tag{24}$$

on each interval of continuity of $\eta^*(x)$, $a^*(x)$.

4. The inequality

$$H(u^*, p^*, \eta, a, 0, 0) = p^*\eta - \lambda_0 a(\tfrac{1}{2}\eta^2 - u^*) - \lambda a \leq H(u^*, p^*, \eta^*, 0, 0) = \sup H = \bar{H} \tag{25}$$

holds for all admissible controls $\eta(x)$ and $a(x)$ on the left side and for $u^*(x)$, $\eta^*(x)$, $a^*(x)$, together with the solution values of the multipliers $p^*(x)$, λ_0, λ on both sides of (25). The constant $\bar{H}$ is the value of the supremum of H guaranteed by the result Eq. (24) since the H function does not depend explicitly on x. Since the boundary values on $u(x)$ are given at both ends of the rod, there are no boundary conditions on $p(x)$ arising from the standard transversality relations.

As in Section 2, the left side of the Maximum-H inequality is replaced by the related function

$$\tilde{H}(a, \eta) = p(x)\eta - a(\tfrac{1}{2}\eta^2 - u(x)) - \lambda a \tag{26}$$

and its supremum is sought with respect to the controls regarding the other quantities as known. There are four steps in the construction of the global optimizer for all values of the parameters a_1, a_2, and δ. First, the value 0 for the nonnegative scalar Lagrange multiplier λ_0 can be ruled out by showing that all multipliers would then be zero; therefore $\lambda_0 = 1$ from here on [Appendix B(1)]. The multiplier function $p(x)$ can then be identified with the nondimensional axial force since the governing Eqs. (23b) and (23c) become the force equilibrium equation and the elasticity law when $\lambda_0 = 1$. The optimal choice of the strain $\eta(x)$ is then p/a for each choice of $a(x)$. Second, completing the calculation of $\sup \tilde{H}$ with respect to $a(x)$ shows that the optimal $a(x)$ must be discontinuous and consist of intervals either at a_1 or at a_2 [Appendix B(2)]. Satisfying the volume constraint tells us that the total length (measure) of the intervals at a_1 and of those at a_2 must be ξ_1 and ξ_2 as given in (21e). Therefore, the optimal design must consist of at least two segments. Third, Eqs. (23a), (23b), and (23c) can be solved for any constant-section segment and the value of H shown to be a constant necessarily in any such segment. These results together with continuity of $u(x)$ and $p(x)$ at the joins between segments are used to prove that no design with alternating segments of two constant sizes can be optimal if there are four or more segments [Appendix B(3)]. This leaves two designs of three segments each together with the two with two segments as the candidate optimizers. Finally, in Appendix B(4), by comparison of the $\bar{H}$ values for the four candidates and their associated multiplier functions $\mu(x)$ and evaluation of the potential energy functional π, one establishes the following.

Proposition. If $|\delta| < \delta_1 = \xi_1/2a_1$, then the three-region design with $a(x) = \{a_1, a_2, a_1\}$ in intervals of lengths $\{a_1\delta + \xi_1/2, \xi_2, -a_1\delta + \xi_1/2\}$ is the sole optimal design. If $\delta \geq \delta_1$, then the two-region design $a(x) = \{a_1, a_2\}$ with lengths necessarily $\{\xi_1, \xi_2\}$ is optimal. If $\delta \leq -\delta_1$, then the two-region design $a(x) = \{a_2, a_1\}$ with lengths necessarily $\{\xi_2, \xi_1\}$ is optimal. An optimal three-region design $\{a_1, a_2, a_1\}$ does not exist for $|\delta| \geq \delta_1$ whereas the candidate three-region design $\{a_2, a_1, a_2\}$ cannot be optimal for any value of δ.

These results agree with those of Fosdick and Royer-Carfagni (1996).

Acknowledgements

I wish to thank Professor Roger Fosdick for many helpful and stimulating discussions related to nonconvex minimization problems. I also thank Professor E. Bruce Lee for guidance in the interpretation of the Maximum Principle.

Appendix A

Calculations for the minimum volume problem

1. The optimal value of $v(x)$ is zero: From the first of Eqs. (13b), the dual function $v(x)$ must be a constant v^*. If that constant is nonzero, choosing η equal to sign (v) will make $\tilde{H}$ in Eq. (17) as large as possible without violating the constraints on η. But if η is a constant other than zero, then [from (13a)] $u(x)$ would be linear in x and both boundary conditions on u could not be met. Thus, zero is the only possible value for $v=v^*$ and η no longer appears explicitly in $\tilde{H}$. Indeed we see that the boundary conditions on u require that the solution $\eta(x)$ have integral zero over the rod length.
2. The value of λ_0 cannot be zero and so can be taken as 1: Since v must be zero then if λ_0 is also zero $\tilde{H}$ would reduce to its last term: $\tilde{H}=-\gamma qa$. This will be maximised when $q > 0$ if $a = 1$ and when $q < 0$ if $a = \hat{a}$. Since $\sup \tilde{H}$ must be constant and $a(x)$ is constant, q must be of one sign and constant. But $q = 0$ is the only constant satisfying the boundary conditions (16), and then $\sup \tilde{H}$ is zero. More, from (13b), (13c) and (13d) we would find $\mu_5=0$, $\mu_1=\mu_2$, $\mu_3=\mu_4$. But $\mu_1=\mu_2$, $\mu_3=\mu_4$ can hold only if all are zero; and so all eight multipliers would be simultaneously zero, which is not allowed. Therefore λ_0 cannot be zero. Once λ_0 is known to be positive, it can be taken as a multiplicative scaling factor for the other multipliers and so for the function H; it suffices to set $\lambda_0=1$.
3. Continuity of the optimal design: Set $v = 0$ and $\lambda_0=1$ in $\tilde{H}$ and in turn set the latter equal to the constant value $\bar{H}$ which is yet to be found. The optimal design must satisfy $a(x)=-\bar{H}/(\gamma q(x)+1)$ and so be continuous on [0, 1] along with u, p, and q. The strain $\eta(x)=p/a$ must also then be continuous.
4. The optimal design for $\gamma \leq 2$: A solution with $a(x)$ constant for the whole interval requires $q(x)$ also constant from $\bar{H}=-a(\gamma q + 1)$. $q = 0$ is the only value that satisfies the boundary conditions on q. If $q = 0$ so is μ_5 from (13b) and, from (13c), $\mu_1-\mu_2=1$. Since these last two μs cannot be nonzero together and one must be nonzero and positive, the only possibility is for $\mu_1=1$, $\mu_2=0$. The optimal choice of $a(x)$ must be the lower bound 1. This is just the solution found above by direct considerations at the start of Section 2. The remainder of the functions p, η, and u can now be found:

   $$p(x) = \mu(x) = \tfrac{1}{2} - x, \quad u(x) = \tfrac{1}{2}x(1 - x).$$

   The limiting value $\gamma=2$ comes from setting $|\eta|=1$ at $x = 0$ and 1. Note that the value of $\bar{H}$ for this solution is -1.
5. The optimal design for $\gamma > 2$: When $\gamma > 2$ one must consider the other possibility that $|\eta|=1$ over parts of (0, 1) so that from (11) $p(x)= \pm a(x)$ there. It follows that $a(x)$ must be proportional to $\exp[-\gamma x]$ where $\eta=+1$ and to $\exp[+\gamma x]$ where $\eta=-1$. Since p is continuous, intervals these two types cannot be adjacent but must be separated by an interval where $a(x)$ is constant and where η changes continuously from $+1$ to -1. The sole zero of $p(x)$ and of $\eta(x)$ must lie in this interval. Moreover, any regions with $\eta=+1$ must occur at the top of the column and any with $\eta=-1$ must be at the bottom.

It is not hard to show that two-region candidates with a constant-section region followed by an $\eta=-1$ region or an $\eta=+1$ region followed by a constant-section region cannot occur. This is done by tracing the $q(x)$ function (necessarily zero if $a(x)$ is constant near either end of the column) and showing that the location of the join between the two regions would be at the other end of the rod such that no variable section could occur. The first candidate then is a three-region design with $\eta = +1$ for $0 < x < \hat{x}$; $a = a_0$ for $\hat{x} < x < \tilde{x}$; and $\eta = -1$ for $\tilde{x} < x < 1$, where the constant value a_0 must be determined (and will turn out to be the minimum value 1). Optimal designs with higher numbers of intervals cannot exist since the continuous functions p and η are monotone decreasing.

It is a straightforward if somewhat tedious task to integrate the equations in each region and from continuity conditions as the join points $\hat{x}, \tilde{x}$ determine the values

$$\hat{x} = \frac{1}{2} - \frac{1}{\gamma}, \quad \tilde{x} = \frac{1}{2} + \frac{1}{\gamma}$$

and the other constants of integration. The resulting optimal design $a^*(x)$ is given in Eq. (18). It is symmetric about the midpoint $x = 1/2$. The remaining quantities $u(x) = \eta(x)$, $p(x)$, and $q(x)$ are

$$u(x) = \begin{cases} x \\ \frac{\gamma}{2}x(1-x) - \frac{(\gamma-2)^2}{8\gamma} \\ 1-x \end{cases}; \quad \eta(x) = \begin{cases} 1 \\ \gamma\left(\frac{1}{2} - x\right) \\ -1 \end{cases};$$

$$p(x) = \begin{cases} a^*(x) \\ \eta(x) \\ -a^*(x) \end{cases}; \quad q(x) = \begin{cases} -\frac{1}{\gamma}(1 - \exp[\gamma x]) \\ -\frac{1}{\gamma}(1 - \exp[\frac{\gamma-2}{2}]) \\ -\frac{1}{\gamma}(1 - \exp[\gamma(1-x)]) \end{cases}.$$

Moreover, the multipliers $\mu_k(x)$ and the constant $\bar{H}$ can be found; the multipliers that must be positive in each interval are indeed positive:

$$0 < x < \hat{x}: \quad \mu_1 = \mu_2 = \mu_3 = 0, \quad \mu_4 = -\mu_5 = \exp\left[\frac{\gamma-2}{2}\right] > 0;$$

$$\hat{x} < x < \tilde{x}: \quad \mu_2 = \mu_3 = \mu_4 = \mu_5 = 0, \quad \mu_1 = \exp\left[\frac{\gamma-2}{2}\right] > 0;$$

$$\tilde{x} < x < 1: \quad \mu_1 = \mu_2 = \mu_4 = 0, \quad \mu_3 = \mu_5 = \exp\left[\frac{\gamma-2}{2}\right] > 0;$$

$$\bar{H} = -\exp\left[\frac{\gamma-2}{2}\right].$$

Note that the value of $\bar{H}$ is less than the value of -1 found for the $\bar{H}$ of Section 2 for the rod of uniform section. Thus the solution found here would not be optimal if the uniform rod satisfied all the constraint conditions for $\gamma > 2$.

This solution exists for all $\gamma > 2$, with the division points $\hat{x}$, $\tilde{x}$ moving in from the ends toward the midpoint as γ approaches infinity. The limitation on the range of γ arises from the upper bound constraint on $a(x)$. For the maximum a at either end of the bar to remain less than $\hat{a}$, we must have $\gamma\hat{x} = (\gamma - 2)/2 < \ln(\hat{a})$ or

$$2 < \gamma < 2 + 2\ \ln(\hat{a})$$

for the solutions with three regions to exist.

We must rule out the possibility of a higher number or regions, with more than one constant-η and constant-a regions alternating on each side of a central one with uniform section. The argument given above for the constant-a design being equal to the minimum section size wherever a and q are constant still holds, however, so that one could not have an exponentially decaying section with $\eta = +1$ to the right of a uniform section at $a = 1$ or an exponentially growing section to the left of such a section without violating the lower bound constraint on $a(x)$. Together with the monotone character of $p(x)$ and $\eta(x)$, this shows that we cannot have more than three regions. Thus we have a unique optimizer in each γ interval where solutions exist with no solution possible for $\gamma > 2 + 2\ \ln(\hat{a})$.

The total volume for this optimal design is

$$\int_0^1 a^*(x)\mathrm{d}x = \frac{2}{\gamma}\ \exp\left[\frac{\gamma}{2} - 1\right]$$

which is greater than one when $\gamma > 2$. For example, if $\hat{a}$ is greater than the number e so that $\gamma = 4$ is an allowable value, one finds that the minimum volume is $e/2$ or nearly 36% higher than the volume for $\gamma \leq 2$.

Appendix B

Calculations for the minimum energy problem

1. The value of λ_0 cannot be zero and so can be taken as 1: If λ_0 were zero, $p(x) = 0$ from Eq. (23c). Suppose λ is not zero. If it were positive (negative) than $\mu_1 - \mu_2$ would need to be positive (negative) from (23d). Since the μs must be nonnegative and cannot be nonzero at the same place, then either $\mu_1 > 0$ or $\mu_2 > 0$ along the whole rod. This would require that either $a = a_1$ or $a = a_2$. But the volume constraint cannot then be met. Therefore λ must be zero and $\mu_1 = \mu_2$. Since ϕ_1 and ϕ_2 cannot be zero together, the only way the μs can be equal is if they are both zero. Then all five multipliers would be zero simultaneously, which is not allowed by the Principle. Thus, λ_0 cannot be zero and can be taken to be 1.
2. The optimal $a(x)$ must be piecewise constant: Maximise

$$\tilde{H}(a, \eta) = p(x)\eta - a(\tfrac{1}{2}\eta^2 - u(x)) - \lambda a \tag{B1}$$

with respect to a, η treating $p(x)$ and $u(x)$ as known. The first and second η derivatives of the function in (B1) are $p - a\eta$ and $-a$. The first derivative vanishes (Eq. (23c) again) at

$$\eta = p/a \tag{B2}$$

while the second derivative is always negative, indicating a relative maximum. Substituting this value of η back into (B1) we obtain the intermediate function H^* of a:

$$H^*(a) = \tilde{H}(a, p/a) = \frac{(p)^2}{2a} + a(u - \lambda). \tag{B3}$$

We now seek the supremum of the latter subject to the inequality constraints on a.

Since the second derivative of H^* with respect to a is the always non-negative expression

$$\frac{\partial^2 H^*}{\partial a^2} = \frac{p^2}{a^3}$$

the supremum of H^* does not occur where the first derivative is zero but at one of the end points $a = a_1$ or $a = a_2$. Since neither can be extended over the whole length and still satisfy the volume constraint, there must be at least two segments in any admissible design consisting of just these two values for $a(x)$. The value or values of x where switching from one size to the other occurs, become quantities that must be found.

To satisfy the volume constraint the total lengths (measures) ξ_1 and ξ_2 at each of the bounds must satisfy

$$\xi_1 = \xi_2 = 1, \quad a_1\xi_1 + a_2\xi_2 = a_0 \tag{B4a}$$

and so be

$$\xi_1 = \frac{a_2 - a_0}{a_2 - a_1}, \quad \xi_2 = \frac{a_2 - a_1}{a_2 - a_1}. \tag{B4b}$$

A useful identity for later work is obtained by substituting the last forms for the ξs back into the harmonic mean identity of Eq. (21e) and rearranging:

$$a_2 + a_1 - a_0 \equiv a_1 a_2. \tag{B4c}$$

3. The solution forms for segments of constant section; proof that no design with four or more regions can be optimal: For any segment of the rod in which $a(x)$ is constant, the solutions $p(x)$ and $u(x)$ to the equations for the elastic problem are linear and quadratic in x, respectively. For $a(x) = a_\alpha$, $\alpha = 1$, 2,

$$p(x) = a_\alpha \eta(x) = p_0 - a_\alpha(x - x_0)$$

$$u(x) = u_0 + \frac{p_0}{a_\alpha}(x - x_0) - \frac{(x - x_0)^2}{2} = \left(\frac{p_0^2}{2a_\alpha^2} + u_0\right) - \frac{p^2(x)}{2a_\alpha^2} = \frac{E_\alpha}{a_\alpha} - \frac{p^2(x)}{2a_\alpha^2} \tag{B5}$$

where x_0 is an arbitrarily chosen point of the segment, p_0 and u_0 are the values of the force and displacement there, and the 'energy constant' E_α has been introduced. It is easily seen that the value of E_α is independent of the choice of x_0. The value of the Hamiltonian function needed in the Maximum Principle inequality will be constant where $a(x)$ is constant:

$$H(u(x), p(x), a_\alpha, \eta(x), 0, 0) = a_\alpha\left(\frac{1}{2}\eta^2(x) + u(x) - \lambda\right) = \frac{p_0^2}{2a_\alpha} + a_\alpha(u_0 - \lambda) = E_\alpha - \lambda a_\alpha. \tag{B6}$$

For a sequence of segments in which α alternates between 1 and 2, all can be made to have the same constant Hamiltonian value $\bar{H}$. Moreover, the value of E_α is the same for all segments where $a = a_\alpha$. Setting the H-function for the two types of regions equal to one another gives

$$\lambda = \frac{E_2 - E_1}{a_2 - a_1},$$

$$\bar{H} = \frac{a_2 E_1 - a_1 E_2}{a_2 - a_1} = \xi_1 E_1 + \xi_2 E_2 - \lambda a_0. \tag{B7}$$

An additional useful form of $\bar{H}$ that leads directly to the result that there cannot be more than three segments in an optimal design is obtained by taking the reference point x_0 for two neighboring segments at the join $x = \tilde{x}$ between them and using the common nodal values $\tilde{p}$ and $\tilde{u}$ there in the 'energy' constants E_α. Then

$$\lambda = \tilde{u} - \frac{\tilde{p}^2}{2a_1 a_2}, \quad \xi_1 E_1 + \xi_2 E_2 = a_0 \tilde{u} + \frac{\tilde{p}^2}{2},$$

$$\bar{H} = \left(1 + \frac{a_0}{a_1 a_2}\right)\frac{\tilde{p}^2}{2} = \frac{(a_1 + a_2)}{2a_1 a_2}\tilde{p}^2 \tag{B8}$$

where the identity of (B4c) has been used in the last line.

From this it follows immediately that no more than three segments are possible in an optimal design. Since $\bar{H}$ must be constant for an optimizer the value $\tilde{p}^2$ must be the same at all joins between segments of different sizes. Since $p(x)$ is a continuous monotone decreasing function, it can take on each of its values only once. Thus, if there is a second join the value there of $p(x)$ must be negative of that at the first join and there can be no further occurrences of the square of that value. It also follows that if there are three segments the value of $p(x)$ at the midpoint of the middle segment must be zero and that the length of that segment will be ξ_α if its size is a_α, with the value $\tilde{p}$ at the top join equal to $a_\alpha \xi_\alpha / 2$.

The forms of the multiplier functions which must be positive if a design is to be optimal are obtained from the equation $\partial H/\partial a = 0$ [Eq. (13)]:

$$\mu_1(x) - \mu_2(x) = \lambda + \tfrac{1}{2}\eta^2(x) - u(x).$$

Therefore, using (B6), where $a = a_1$, μ_2 must be 0 and

$$\mu_1(x) = \lambda - \frac{E_1}{a_1} + \frac{p^2(x)}{a_1^2} + \frac{p^2(x)}{a_1^2} = \frac{p^2(x)}{a_1^2} - \frac{\bar{H}}{a_1} \tag{B9a}$$

must be positive for optimality and, where $a = a_2$, $\mu_1 = 0$ and

$$\mu_2(x) = -\lambda + \frac{E_2}{a_2} - \frac{p^2(x)}{a_2^2} = \frac{\bar{H}}{a_2} - \frac{p^2(x)}{a_2^2} \tag{B9b}$$

must be positive for optimality.

The contribution of a segment with $a(x) = a_\alpha$ to the additive functional π is also easily calculated. Let $\{x^-, x^+\}$ denote the endpoints of the segment and set $x^+ - x^- = \zeta$, the length of the segment. Then the potential energy of the segment is

$$\int_{x^-}^{x^+} a_\alpha \left(\frac{1}{2}\eta^2(x) - u(x)\right) dx = \int_{x^-}^{x^+} \left(\frac{p^2(x)}{a_\alpha} - E_\alpha\right) dx = \frac{p^3(x^-) - p^3(x^+)}{3a_\alpha^2} - E_\alpha \zeta. \tag{B10}$$

4. The four candidate optimizers; selection of the global optimizer: Next list the candidates. Choose the reference points x_0 of Eqs. (B5) for the top and bottom segments to be at $x = 0$ and $x = 1$, respectively, and for the middle segment in the three-region designs at its midpoint $\hat{x}$. Call the force values at the top and bottom of the rod p^* and p^{**}, respectively. For the three-region designs, the midpoint strain $\hat{p} = 0$; see the discussion after Eq. (B8). Call the displacement there $\hat{u}$. The four different candidates will be distinguished where necessary by Roman numeral superscripts I, II, III, and IV, where I is the two-region candidate with $a(x) = \{a_1, a_2\}$, II is the two-region candidate with $a(x) = \{a_2, a_1\}$, III is the three-region candidate with $a(x) = \{a_1, a_2, a_1\}$, and IV is the three-region candidate with $a(x) = \{a_2, a_1, a_2\}$.

The solutions for Cases I and II exist for all values of the end displacement δ. For I, the results of the calculation for the end forces and for the displacement and force at the join $\tilde{x} = \xi_1$ are

$$p^* = \delta + \hat{\delta} + \frac{\xi_2}{a_2} a_0, \quad p^{*}* = \delta + \hat{\delta} - \frac{\xi_1}{a_1} a_0,$$

$$\tilde{u} = u(\tilde{x}) = \frac{\xi_1}{a_1}\left(\delta - \hat{\delta} + \frac{a_1 \xi_1}{2}\right), \quad \tilde{p} = p(\tilde{x}) = \delta - \hat{\delta} \tag{B11}$$

where

$$\hat{\delta} = \frac{\xi_1 - \xi_2}{2}. \tag{B12}$$

To derive these results the identities Eqs. (21e, B4) involving the as and ξs have been used, as they must often be to simplify the calculations that follow. We also need the values of $\bar{H}$, $\mu(x)$, and π quantities for Case I which are obtained using the general results in Eqs. (B8)–(B10):

$$\bar{H}^{\mathrm{I}}(\delta) = \frac{(a_2 + a_1)}{2 a_1 a_2} (\delta - \hat{\delta})^2 \tag{B13a}$$

$$\mu^{\mathrm{I}}(x) = \begin{cases} \dfrac{(p^* - a_1 x)^2}{a_1^2} - \dfrac{\bar{H}^{\mathrm{I}}}{a_1} \\ \dfrac{\bar{H}^{\mathrm{I}}}{a_2} - \dfrac{(p^{*}* - a_2 (x-1))^2}{a_2^2} \end{cases} \tag{B13b}$$

$$\pi^{\mathrm{I}}(\delta; a_1 a_2) = \frac{1}{2}(\delta - \hat{\delta})^2 - a_2 \xi_2 (\delta - \hat{\delta}) + \frac{a_1 \xi_1^3 + a_2 \xi_2^3}{3} - \frac{a_0 \xi_1^2}{2}. \tag{B13c}$$

For Case II, which also exists for all values of δ, the join point is at $\tilde{x} = \xi_2$. It is easy to show that the results for Case II may be computed from those for Case I by making the substitution of $1-x$ for x and $-\delta$ for δ. One finds

$$u^{\mathrm{II}}(x;\delta) = u^{\mathrm{I}}(1 - x; -\delta) + \delta. \tag{B14}$$

The $\bar{H}$, $\mu(x)$, and π quantities for Case II are

$$\bar{H}^{\mathrm{II}}(\delta) = \bar{H}^{\mathrm{I}}(-\delta) = \frac{(a_2 + a_1)}{2 a_1 a_2} (\delta + \hat{\delta})^2 \tag{B15a}$$

$$\mu^{\mathrm{II}}(x,\,\delta)=\mu^{\mathrm{I}}(1-x,\ -\delta)=\begin{cases}\dfrac{\bar{H}^{\mathrm{II}}}{a_2}-\dfrac{(p^{\mathrm{II}}(x))^2}{a_2^2}\\[2ex]\dfrac{(p^{\mathrm{II}}(x))^2}{a_1^2}-\dfrac{\bar{H}^{\mathrm{II}}}{a_1}\end{cases}\tag{B15b}$$

$$\pi^{\mathrm{II}}(\delta;a_1,\,a_2)=\ \pi^{\mathrm{I}}(-\delta;a_1a_2)-a_0\delta=\frac{1}{2}(\delta+\hat{\delta})^2-a_1\xi_1\delta-\frac{a_1\xi_1^3+a_2\xi_2^3}{6}.\tag{B15c}$$

To construct the candidate extremals with three regions, first find the lengths of the top and bottom segments. The length ζ_2 of the middle segment is necessarily ξ_2 for Case III and ξ_1 for Case IV. Call the length of the top segment ζ_1 and that of the bottom ζ_3; $\zeta_1+\zeta_3$ equals ξ_1 for Case III and ξ_2 for Case IV. The switching points between segments are at $x=\tilde{x}_1=\zeta_1$ and $x=\tilde{x}_2=1-\zeta_3$. From the discussion after (B8) the values of p and u at the switching points are

$$\tilde{p}_1=-\tilde{p}_2=\frac{a_2\xi_2}{2}\quad\text{(Case III)};\quad=\frac{a_1\xi_1}{2}\quad\text{(Case IV)};$$

$$\tilde{u}_2=\tilde{u}_1=\hat{u}-\frac{\xi_2^2}{8}\quad\text{(Case III)};\quad=\hat{u}-\frac{\xi_1^2}{8}\quad\text{(Case IV)}.\tag{B16}$$

Now formulate the force relations between the ends of each of the top and bottom segments and the displacement boundary conditions at $x = 0$ and $x = 1$. For Case III, using the equality of the Es for the top and bottom segments in an optimal design, one finds

$$\zeta_1=\frac{\xi_1}{2}+\frac{a_1}{a_0}\delta,\quad \zeta_3=\frac{\xi_1}{2}-\frac{a_1}{a_0}\delta\tag{B17a}$$

with a similar calculation for Case IV resulting in

$$\zeta_1=\frac{\xi_2}{2}+\frac{a_2}{a_0}\delta,\quad \zeta_3=\frac{\xi_2}{2}-\frac{a_2}{a_0}\delta.\tag{B17b}$$

From these we see that three-region candidates can exist only for a range of values of δ. The vanishing of the ζs gives III and IV existing only for

$$|\ \delta\ |<\delta_1=\frac{a_0\xi_1}{2a_1}\quad\text{(Case III)}$$

$$|\ \delta\ |<\delta_2=\frac{a_0\xi_2}{2a_2}\quad\text{(Case IV)}.\tag{B18}$$

Now construct the solutions for III and IV. For III, the results for end forces and the displacement at $\hat{x}$ will be

$$p^*=\frac{a_0}{2}+\frac{a_1^2\delta}{a_0},\quad p^**=-\frac{a_0}{2}+\frac{a_1^2\delta}{a_0}\tag{B19}$$

$$\hat{u} = \frac{a_1^2}{2a_0^2}\delta^2 + \frac{1}{2}\delta + \frac{1}{8}\left(\xi_2^2 - \xi_1^2\right) + \frac{a_0\xi_1}{4a_1}. \tag{B20}$$

The $\bar{H}$, $\mu(x)$, and π quantities for Case III are:

$$\bar{H}^{\mathrm{III}} = \frac{a_2(a_2 + a_1)\zeta_2^2}{8a_1} \tag{B21}$$

$$\mu^{\mathrm{III}}(x) = \begin{cases} \dfrac{(p^{\mathrm{III}}(x))^2}{a_1^2} - \dfrac{\bar{H}^{\mathrm{III}}}{a_1} \\ \dfrac{\bar{H}^{\mathrm{III}}}{a_2} - (x - \hat{x})^2 \\ \dfrac{(p^{\mathrm{III}}(x))^2}{a_1^2} - \dfrac{\bar{H}^{\mathrm{III}}}{a_1} \end{cases} \tag{B22}$$

$$\pi(\delta; a_1, a_2) = \pi^{\mathrm{III}}(0; a_1, a_2) - \frac{a_0}{2}\delta + \frac{a_1^2}{2a_0}\delta^2 \tag{B23a}$$

where

$$\pi^{\mathrm{III}}(0; a_1 a_2) = \frac{a_1\xi_1^3 + a_2\xi_2^3}{12} - \frac{a_0\xi_1^2}{8} - \frac{a_2^2\xi_2^2}{8}. \tag{B23b}$$

For Case IV, the solution is obtained by interchanging subscripts 1 and 2 on the as and ξs in the Case III solution. The first result is that IV is not the global optimizer at any value of δ. This is found by evaluating the multiplier function $\mu^{\mathrm{IV}}(x)$ at the midpoint $\hat{x}^{\mathrm{IV}} = \frac{1}{2} + (a_2/a_0)\delta$ and seeing that it must always be negative:

$$\mu^{\mathrm{IV}}(x_M^{\mathrm{IV}}) = -\frac{\bar{H}^{\mathrm{IV}}}{a_1} = -\frac{(a_2 + a_1)}{8a_2}\xi_1^2. \tag{B24}$$

The second result is that the multiplier function $\mu^{\mathrm{III}}(x)$ is positive in each of its intervals of continuity for all δ for which III exists. This follows easily by considering the graph of $\mu^{\mathrm{III}}(x)$ which consists of three parabolic pieces, opening upward for x in the top and bottom regions and downward in the middle. The minimum values for each part are the values at the joins and these are all positive.

We must next compare the two-region candidates to III. From differences of $\bar{H}$ values it is not hard to show that I has a higher $\bar{H}$ value than III when $\delta > \delta_1$ (where in fact candidate III no longer exists) and similarly II has a higher values than III when $\delta < \delta_1$. In $|\delta| < \delta_1$, III has the highest $\bar{H}$ value of the three if $\hat{\delta} = (\xi_1 - \xi_2)/2 < 0$. If $\hat{\delta} > 0$, an ambiguity arises that cannot be settled by an appeal only to the positivity of the corresponding $\mu(x)$ functions. A more detailed argument based on the supremum inequality itself could be given but is not needed since the values for the energy functions can be compared directly. Some algebra will show that

$$\pi^{\mathrm{I}}(\delta; a_1, a_2) - \pi^{\mathrm{III}}(\delta; a_1, a_2) = \left(\frac{1}{2} - \frac{a_1^2}{2a_0}\right)(\delta - \delta_1)^2 \geq 0. \tag{B25}$$

$$\pi^{\mathrm{II}}(\delta;a_1, a_2) - \pi^{\mathrm{III}}(\delta;a_1, a_2) = \left(\frac{1}{2} - \frac{a_1^2}{2a_0}\right)(\delta + \delta_1)^2 \geq 0. \tag{B26}$$

$$\pi^{\mathrm{II}}(\delta;a_1, a_2) - \pi^{\mathrm{I}}(\delta;a_1, a_2) = 2\delta_1\delta. \tag{B27}$$

Therefore, III has lower energy than I or II whenever III exists; Case I has lower energy than II for $\delta > 0$ and II than I when $\delta < 0$. This completes the proof of the proposition at the end of Section 3 for all ranges of δ.

References

Fosdick, R., Royer-Carfagni, G., 1996. The static state of a two-phase solid mixture in a stressed elastic bar. Int. J. Solids and Structures 33, 2267–2281.

Fosdick, R., Royer-Carfagni, G., Weinberger, H., 1996. A note on the optimal state of a binary solid mixture in a stressed elastic bar. Meccanica 31, 519–525.

Hestenes, M., 1980. Calculus of Variations and Optimal Control Theory. Robert E Krieger Publishing Co, Huntington, New York.

Pontryagin, L., Boltyanskii, V., Gamkrelidze, R.V., Mishchenko, E.F., 1962. In: Neustadt, L. (Ed.), The Mathematical Theory of Optimal Processes. Interscience, New York.

Velte, W., Villaggio, P., 1982. Are the optimum problems in structural design well posed? . Arch. Rat. Mech. Anal. 78, 199–211.